An Easy Path to Convex Analysis and Applications

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Boris S. Mordukhovich and Nguyen Mau Nam
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An Easy Path to Convex Analysis and Applications

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ABSTRACT

Convex optimization has an increasing impact on many areas of mathematics, applied sciences, and practical applications. It is now being taught at many universities and being used by researchers of different fields. As convex analysis is the mathematical foundation for convex optimization, having deep knowledge of convex analysis helps students and researchers apply its tools more effectively. The main goal of this book is to provide an easy access to the most fundamental parts of convex analysis and its applications to optimization. Modern techniques of variational analysis are employed to clarify and simplify some basic proofs in convex analysis and build the theory of generalized differentiation for convex functions and sets in finite dimensions. We also present new applications of convex analysis to location problems in connection with many interesting geometric problems such as the Fermat-Torricelli problem, the Heron problem, the Sylvester problem, and their generalizations. Of course, we do not expect to touch every aspect of convex analysis, but the book consists of sufficient material for a first course on this subject. It can also serve as supplemental reading material for a course on convex optimization and applications.

KEYWORDS

Affine set, Carathéodory theorem, convex function, convex set, directional derivative, distance function, Fenchel conjugate, Fermat-Torricelli problem, generalized differentiation, Helly theorem, minimal time function, Nash equilibrium, normal cone, Radon theorem, optimal value function, optimization, smallest enclosing ball problem, set-valued mapping, subdifferential, subgradient, subgradient algorithm, support function, Weiszfeld algorithm

The first author dedicates this book to the loving memory of his father
SHOLIM MORDUKHOVICH *(1924–1993),*
a kind man and a brave warrior.

The second author dedicates this book to the memory of his parents.

Contents

Preface

Some geometric properties of convex sets and, to a lesser extent, of convex functions had been studied before the 1960s by many outstanding mathematicians, first of all by Hermann Minkowski and Werner Fenchel. At the beginning of the 1960s convex analysis was greatly developed in the works of R. Tyrrell Rockafellar and Jean-Jacques Moreau who initiated a systematic study of this new field. As a fundamental part of variational analysis, convex analysis contains a generalized differentiation theory that can be used to study a large class of mathematical models with no differentiability assumptions imposed on their initial data. The importance of convex analysis for many applications in which convex optimization is the first to name has been well recognized. The presence of convexity makes it possible not only to comprehensively investigate qualitative properties of optimal solutions and derive necessary and sufficient conditions for optimality but also to develop effective numerical algorithms to solve convex optimization problems, even with nondifferentiable data. Convex analysis and optimization have an increasing impact on many areas of mathematics and applications including control systems, estimation and signal processing, communications and networks, electronic circuit design, data analysis and modeling, statistics, economics and finance, etc.

There are several fundamental books devoted to different aspects of convex analysis and optimization. Among them we mention "Convex Analysis" by Rockafellar [26], "Convex Analysis and Minimization Algorithms" (in two volumes) by Hiriart-Urruty and Lemaréchal [8] and its abridge version [9], "Convex Analysis and Nonlinear Optimization" by Borwein and Lewis [4], "Introductory Lectures on Convex Optimization" by Nesterov [21], and "Convex Optimization" by Boyd and Vandenberghe [3] as well as other books listed in the bibliography below.

In this big picture of convex analysis and optimization, our book serves as a bridge for students and researchers who have just started using convex analysis to reach deeper topics in the field. We give detailed proofs for most of the results presented in the book and also include many figures and exercises for better understanding the material. The powerful geometric approach developed in modern variational analysis is adopted and simplified in the convex case in order to provide the reader with an easy path to access generalized differentiation of convex objects in finite dimensions. In this way, the book also serves as a starting point for the interested reader to continue the study of nonconvex variational analysis and applications. It can be of interest from this viewpoint to experts in convex and variational analysis. Finally, the application part of this book not only concerns the classical topics of convex optimization related to optimality conditions and subgradient algorithms but also presents some recent while easily understandable qualitative and numerical results for important location problems.

The book consists of four chapters and is organized as follows. In Chapter 1 we study fundamental properties of convex sets and functions while paying particular attention to classes of convex functions important in optimization. Chapter 2 is mainly devoted to developing basic calculus rules for normals to convex sets and subgradients of convex functions that are in the mainstream of convex theory. Chapter 3 concerns some additional topics of convex analysis that are largely used in applications. Chapter 4 is fully devoted to applications of basic results of convex analysis to problems of convex optimization and selected location problems considered from both qualitative and numerical viewpoints. Finally, we present at the end of the book Solutions and Hints to selected exercises.

Exercises are given at the end of each chapter while figures and examples are provided throughout the whole text. The list of references contains books and selected papers, which are closely related to the topics considered in the book and may be helpful to the reader for advanced studies of convex analysis, its applications, and further extensions.

Since only elementary knowledge in linear algebra and basic calculus is required, this book can be used as a textbook for both undergraduate and graduate level courses in convex analysis and its applications. In fact, the authors have used these lecture notes for teaching such classes in their universities as well as while visiting some other schools. We hope that the book will make convex analysis more accessible to large groups of undergraduate and graduate students, researchers in different disciplines, and practitioners.

Boris S. Mordukhovich and Nguyen Mau Nam
December 2013

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List of Symbols

$\mathbb{R}$	the real numbers
$\overline{\mathbb{R}} = (-\infty, \infty]$	the extended real line
$\mathbb{R}_+$	the nonnegative real numbers
$\mathbb{R}_>$	the positive real numbers
$\mathbb{N}$	the positive integers
$\text{co}\,\Omega$	convex hull of Ω
$\text{aff}\,\Omega$	affine hull of Ω
$\text{int}\,\Omega$	interior of Ω
$\text{ri}\,\Omega$	relative interior of Ω
$\text{span}\,\Omega$	linear subspace generated by Ω
$\text{cone}\,\Omega$	cone generated by Ω
K_Ω	convex cone generated by Ω
$\dim \Omega$	dimension of Ω
$\overline{\Omega}$	closure of Ω
$\text{bd}\,\Omega$	boundary of Ω
$I\!B$	closed unit ball
$I\!B(\bar{x}; r)$	closed ball with center $\bar{x}$ and radius r
$\text{dom}\, f$	domain of f
$\text{epi}\, f$	epigraph of f
$\text{gph}\, F$	graph of mapping F
$\mathcal{L}[a, b]$	line connecting a and b
$d(x; \Omega)$	distance from x to Ω
$\Pi(x; \Omega)$	projection of x to Ω
$\langle x, y \rangle$	inner product of x and y
A^*	adjoint/transpose of linear mapping/matrix A
$N(\bar{x}; \Omega)$	normal cone to Ω at $\bar{x}$
$\ker A$	kernel of linear mapping A
$D^*F(\bar{x}, \bar{y})$	coderivative to F at $(\bar{x}, \bar{y})$
$T(\bar{x}; \Omega)$	tangent cone to Ω at $\bar{x}$
F_∞	horizon/asymptotic cone of F
$\mathcal{T}^F_\Omega$	minimal time function defined by constant dynamic F and target set Ω
$L(x, \lambda)$	Lagrangian
ρ_F	Minkowski gauge of F

$\Pi_F(\cdot;\Omega)$	generalized projection defined by the minimal time function
$G_F(\bar{x},t)$	generalized ball defined by dynamic F with center $\bar{x}$ and radius t