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Boris S. Mordukhovich

Second-Order Variational Analysis in Optimization, Variational Stability, and Control

Theory, Algorithms, Applications

Springer

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To My Dear Teachers and Friends

YURII BOGDANOV (1920-1987)

RAFAIL GABASOV (1935–2020)

BORIS POLYAK (1935–2023)

TERRY ROCKAFELLAR (born in 1935)

VLADIMIR TIKHOMIROV (born in 1934)

Preface

Variational analysis is a well-recognized area of mathematics with a great many applications to optimization, control, equilibria, stability, machine learning, statistics, as well as to practical models of science and technology. The underlying feature of variational analysis is the broad usage of *generalized differentiation*, which allows us to deal with nonsmooth functions, sets with nonsmooth boundaries, and set-valued mappings. Such objects naturally and frequently appear not only in problems with nonsmooth initial data but mainly due to employing *variational/extremal principles* and *techniques*.

First-order variational analysis is based on (first-order) *generalized derivatives* and *subgradients* of nondifferentiable functions and associated constructions for sets and mappings. Starting with convex analysis, a lot has been done in first-order variational theory and abundant applications; see, e.g., the books [15, 23, 31, 46, 48, 50, 58, 76, 77, 87, 89, 101, 110, 114, 158, 162, 163, 166, 169, 177, 188, 221, 227, 228, 230, 233, 234, 268, 281, 287, 303, 313, 317, 319, 326, 330, 338, 341, 350] with the extensive bibliographies therein.

The situation is different for *second-order variational analysis*. Despite the growing number of strong theoretical developments and a variety of impressive applications documented in numerous journal publications, we present now for the reader's consideration the *first book* entirely devoted to this subject. There exist several *second-order generalized differential* constructions, which are successfully used in second-order variational analysis and its applications. The major ones are reflected in the book, while our main attention is paid to the *second-order subdifferentials* (known also as *generalized Hessians*) of extended-real-valued functions introduced by the author in the early 1990s and then strongly developed and applied by many researchers over the years. The reader can find more discussions, historical remarks, and references in the commentary sections of the book.

The book provides a detailed study of second-order subdifferentials, their *calculus* and efficient *evaluations/calculations* for major classes of functions encountered in problems of variational analysis, optimization, etc. in finite-dimensional and infinite-dimensional spaces. These results and advanced variational techniques turn out to be instrumental in characterizing fundamental notions of *variational stability* of solutions for diverse classes of problems in optimization and optimal control, the study of *variational convexity* of extended-real-valued functions and their specifications, and *variational sufficiency* in optimization among other topics considered in the book. We also include in the book quite recent applications of the second-order subdifferentials, married to the achieved characterizations of variational stability and related concepts, to the design and justification of *second-order numerical algorithms* for solving various classes of optimization problems, nonsmooth equations, and subgradient inclusions. The developed algorithms are *numerically* implemented to solve some *practical models* taken from machine learning, statistics, imaging processes, and other applied areas.

The book consists of *nine* interrelated chapters. To make it *self-contained*, we include in *Chapter 1* some *preliminaries* from first-order variational analysis and generalized differentiation with detailed references. Then this chapter presents, with full proofs, major constructions and properties of *second-order generalized differentiation* and relationships between them, as well as the fundamental notions of *monotonicity*, *prox-regularity*, and *Moreau envelopes*.

Chapter 2 is devoted to *comprehensive calculus rules* for second-order subdifferentials and their partial counterparts in finite and infinite dimensions. The next *Chapter 3* contains *explicit evaluations* and *precise calculations* of second-order subdifferentials for some major classes of extended-real-valued functions and constraint systems. The results of these two chapters play a highly important role in the subsequent developments and applications.

In *Chapter 4*, we start the second-order study of *variational stability* of local optimal solutions and first-order variational systems governed by subgradient inclusions, variational inequalities, and generalized equations. Our main attention here is paid to *robust Lipschitzian stability* of solution mappings for parametric variational systems and *tilt stability* of local minimizers in *general optimization* frameworks as well as infinite-dimensional and finite-dimensional settings of *nonlinear programming*. This chapter also contains a detailed study of related notions of *metric regularity* and *subregularity* for subgradient mappings together with their *strong* counterparts.

Chapter 5 continues the second-order study of variational stability while concentrating now on the notions of *full stability* for *local optimal solutions* to optimization problems in infinite-dimensional and finite-dimensional spaces. We designate here the two different notions of variational stability: *Lipschitzian full stability* and *Hölderian full stability*. Complete *characterizations* of these notions are obtained via second-order growth and second-order subdifferential conditions in general frameworks of extended-real-valued and constrained optimization with explicit specifications of the obtained results for particular classes of constrained optimization problems with *polyhedral* and *nonpolyhedral* structures. Furthermore, close relationships between full stability of local minimizers and *strong stability* of the corresponding Karush–Kuhn–Tucker systems are established under certain nondegeneracy conditions.

Chapter 6 is also devoted to *variational full stability* but of different objects in comparison with the previous chapter, namely, for solution mappings associated with *parametric variational systems* (PVSs) and their specifications. Nevertheless, we keep the same names of *Lipschitzian* and *Hölderian full stability* for PVSs as for local minimizers in Chapter 5, since the obtained characterizations reveal close relationships between these notions under appropriate reductions. A crucial role in the study of full stability for PVSs is played by *local maximal monotonicity* and its *strong* counterpart for set-valued mappings in Hilbert spaces that are of their undoubted own interest. The established *coderivative characterizations* of these notions provide basic ingredients for deriving *second-order characterizations* of full stability for PVSs and their specifications in both finite-dimensional and infinite-dimensional settings.

Chapter 7 addresses *optimal control* of *elliptic PDE systems*. The main goal here is to study *variational full stability* of optimal solutions under various perturbations of problem data. The developed approaches to the study and characterizations of both Lipschitzian and Hölderian full stability for the elliptic PDEs are based on their reductions to problems of *polyhedric programming* in Hilbert spaces with the usage of tools of second-order generalized differentiation married to core PDE theory. It is revealed in this way that both full stability notions are *equivalent* to each other with admitting explicit second-order characterizations. Unfortunately, the size of this book does not allow us to include striking applications of second-order variational analysis to *optimal control* of new classes of dynamical systems governed by *sweeping processes*. We have briefly discussed these topics in the text with giving appropriate references while planning to write a separate book on controlled sweeping processes and related issues in the near future.

Chapter 8 concerns novel concepts of *variational convexity* of extended-real-valued functions and *variational sufficiency* in optimization. To study variational convexity and its *strong* counterpart for functions on Banach spaces, we first revisit *local maximal monotonicity* and *strong maximal monotonicity* but now in general *Banach spaces* that exhibit some significant features in comparison with the Hilbert space versions of Chapter 6. The main result on local maximal monotonicity obtained here is the *resolvent characterization*, which can be viewed as a *local* version of the celebrated *Minty theorem* while being new even in finite dimensions. Using these developments allows us to establish close relationships and equivalences between *variational convexity* of functions and *local convexity* of their *Moreau envelopes* as well as local maximal monotonicity of the associated subgradient mappings. Similar results are obtained for *strong* variational convexity, which happens to be related to *tilt stability* of local minimizers under appropriate geometric assumptions on Banach spaces. The achieved results on variational and strong variational convexity induce the corresponding developments on *variational sufficiency* and *strong variational sufficiency* for local optimality in problems of *composite optimization* with explicit specifications in *nonlinear programming*.

The final *Chapter 9* is, to some extent, the *quintessence* of the book, where theoretical developments and calculations given in the previous chapters are applied to *second-order numerical variational analysis* containing the design, justification, and implementation of numerical algorithms to solve optimization-related problems by using second-order subdifferentials. We present several *generalized Newtonian algorithms* exhibiting local and global convergence with linear, superlinear, and quadratic convergence rates, and then present further applications to solving practical models.

There are extensive *exercises* and *commentaries* at the end of each chapter. Exercises play a significant role in the book. On one hand, they allow the reader to delve deeply into the exciting area of second-order variational analysis and confirm his/her strong understanding of the presented material. On the other hand, some exercises contain results that are not proved while used in the book, with hints and references to the original sources.

For the reader's convenience, we provide the *list of statements* presented in the book, *glossary of notation* and *acronyms*, and detailed *subject index*, which would allow the reader to quickly and easily find the item of interest.

The author hopes that the book will be useful for diverse groups of readers. First of all, it addresses senior and young researchers in the areas of nonlinear, variational and convex analysis, optimization and equilibria, systems control, and their numerous applications to engineering, economics, mechanics, machine learning, statistics, and other branches of applied sciences and technology. Graduate students in all these areas, as well as in real, functional and applied analysis, ordinary and partial differential equations, numerical methods, etc. will gain a great deal from studying this book.

Parts of the book have been used by the author in teaching graduate classes at Wayne State University and other schools and research centers worldwide which he visited during the recent years. Very useful feedback coming from students and all interested listeners is incorporated in the text and exercises.

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